by simply giving the $I=0$ member β_0 of the triplet a different mass than the $I=\frac{1}{2}$ members β_1 and β_2. An additional symmetry-breaking interaction Lagrangian is also needed. Furthermore, in any model which has only one triplet it is difficult to understand why there should be nine approximately degenerate spin-1 meson multiplets while there exist only eight approximately degenerate pseudoscalar meson states. For this reason, it appears that the special model discussed in Sec. II is a more realistic one.

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Explicit Construction of Asymptotic Fields*

EMIL KAZES
Department of Physics, The Pennsylvania State University, University Park, Pennsylvania
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Examination of a separable potential model in field theory, when the interaction is attractive enough to produce bound states, shows that the $t=\pm\infty$ limits of the Heisenberg fields do not always have a particle interpretation but are superpositions of eigenfields. In this model the commutators of the in-fields of particles that are well enough localized to have a finite interaction energy are operators.

I. INTRODUCTION

ASYMPTOTIC fields play a central role in the axiomatic formulation of field theory given by Lehmann, Symanzik, and Zimmermann[1] and in many discussions of the analyticity of the S matrix based on their work. The properties of asymptotic fields have been extensively examined by Zimmermann, Haag, Nishigima, and Ruelle.[2] In order to provide an illustrative example that displays the Heisenberg fields for large times, the infields and their interrelation, we examined a separable potential model in field theory.[3] Within the framework of this model it is shown that: (a) The limits implied in the formal definition of infields[2] exist only after taking their matrix elements. (b) When the interaction is attractive enough to produce bound states, the Heisenberg field of a particle of momentum $\mathbf{k}$ has two terms which oscillate respectively with frequencies $\omega(k)$ and $\mu_n(<\mu)$, as $t\to\pm\infty$. The first term has the usual particle interpretation and reproduces the scattering states, whereas the second term cannot be interpreted as a particle since its energy is below the continuum. The latter term consists of an infinite product of fields and vanishes throughout a subspace that is free of heavy mesons (the target). (c) The commutator of the in-fields of those particles which are well enough localized to have a finite interaction energy is an operator.

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[1] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento 1, 425 (1955).

[2] W. Zimmermann, Nuovo Cimento **10**, 597 (1958); R. Haag, Phys. Rev. **112**, 669 (1958); K. Nishigima, *ibid.* **111**, 995 (1958); K. W. Brenig and R. Haag, Fortschr. Physik **7**, 183 (1959); D. Ruelle, Helv. Phys. Acta **35**, 17 (1962).

[3] For other models see, H. Ezawa, Ann. Phys. (N. Y.) **24**, 46 (1963); Y. Kato and N. Mugibayaski, Progr. Theoret. Phys. (Kyoto) **30**, 103 (1963).

II. SOLUTION OF THE EQUATIONS OF MOTION

The Hamiltonian for a light boson that interacts via a separable potential with a static boson of mass M is

$$H=H_0+\lambda\varphi^\dagger\varphi G^\dagger G, \tag{1}$$

where

$$H_0=M\varphi^\dagger\varphi+\int \mathbf{dk}\,\omega(k)a^\dagger(k)a(k),$$

$$G=\int f(k)a(k)\mathbf{dk}, \tag{2}$$

$$[a(k),a^\dagger(k')]=\delta(k-k'),$$
$$[\varphi,\varphi^\dagger]=1,\ \omega(k)=(\mu^2+k^2)^{1/2}.$$

$a^\dagger(k)$ and $\varphi^\dagger$ are creation operators for a light boson of momentum k energy $\omega(k)$ and a static boson of mass M, respectively. From Eqs. (1) and (2)

$$[H,a^\dagger(k)]=\omega(k)a^\dagger(k)+\lambda\varphi^\dagger\varphi f(k)G^\dagger. \tag{3}$$

In terms of the quantities defined above, the Heisenberg fields are

$$\begin{aligned} e^{iHt}a^\dagger(k)e^{-iHt}&=a^\dagger(k,t),\\ e^{iHt}\varphi^\dagger e^{-iHt}&=\varphi^\dagger(t),\\ G(t)&=\int f(k)a(k,t)\mathbf{dk}. \end{aligned} \tag{4}$$

Since $\varphi^\dagger\varphi$ is a constant of the motion it follows from Eqs. (3) and (4) that

$$-i(d/dt)a^\dagger(k,t)=\omega(k)a^\dagger(k,t)+\lambda\varphi^\dagger\varphi f(k)G^\dagger(t). \tag{5}$$

This is a linear equation in $a^\dagger(k,t)$ that can be solved by taking its Fourier transform by setting

$$a^\dagger(k,t)=\int \tilde{a}^\dagger(k,\omega)e^{i\omega t}d\omega \tag{6}$$

in Eq. (5). This gives

$$\tilde{a}^\dagger(k,\omega)=\frac{1}{\omega-\omega(k)-i\eta}\lambda\varphi^\dagger\varphi f(k)\tilde{G}^\dagger(\omega)+l(k)\delta(\omega(k)-\omega), \tag{7}$$

where $\eta\to 0+$ following the substitution of Eq. (7) in (6). Since

$$\int \tilde{a}^\dagger(k,\omega)f(k)\mathbf{dk}=\tilde{G}^\dagger(\omega),$$

it follows from Eq. (7) that

$$D^-(\omega)\tilde{G}^\dagger(\omega)=\theta(\omega-\mu)\int f(k)l(k)\delta(\omega(k)-\omega)\mathbf{dk}, \tag{8}$$

with

$$D^-(\omega)=1+\lambda\varphi^\dagger\varphi\int\frac{f(k)^2\mathbf{dk}}{\omega(k)-\omega+i\eta}. \tag{9}$$

Substituting in Eq. (6) with (7) yields

$$l(k)=a^\dagger(k)-\lambda\varphi^\dagger\varphi f(k)\int_{-\infty}^{+\infty}\frac{\tilde{G}^\dagger(\omega)}{\omega-\omega(k)-i\eta}d\omega; \tag{10}$$

this combined with Eq. (8) gives

$$D^-(\omega)\tilde{G}^\dagger(\omega)=\theta(\omega-\mu)\bigg[\int f(k)a^\dagger(k)\delta(\omega(k)-\omega)\mathbf{dk}-\lambda\varphi^\dagger\varphi\int\mathbf{dk}f(k)^2\delta(\omega(k)-\omega)\int\frac{\tilde{G}^\dagger(\omega')d\omega'}{\omega'-i\eta-\omega(k)}\bigg]. \tag{11}$$

Defining

$$D(z)=1+\lambda\varphi^\dagger\varphi\int\frac{f(k)^2\mathbf{dk}}{\omega(k)-z},\qquad \mathcal{L}(z)=\int_{-\infty}^{+\infty}\frac{d\omega}{\omega-z}\tilde{G}^\dagger(\omega), \tag{12}$$

and indicating the limits of these functions from above and below the real axis as $D^\pm(\omega)$, $\mathcal{L}(\omega)$, it follows from Eq. (11) that

$$D^+(\omega)\mathcal{L}^+(\omega)-D^-(\omega)\mathcal{L}^-(\omega)=2\pi i\theta(\omega-\mu)\int\mathbf{dk}f(k)a^\dagger(k)\delta(\omega(k)-\omega). \tag{13}$$

Since $D(z\to\infty)=1$, $\mathcal{L}(z\to\infty)=-(1/z)G^\dagger$, from Eq. (12), the solution of Eq. (13) is

$$D(z)\mathcal{L}(z)=\int\mathbf{dk}\frac{f(k)a^\dagger(k)}{\omega(k)-z}. \tag{14}$$

Combining this with Eq. (10) yields

$$l(k)=a^\dagger(k)-\lambda\varphi^\dagger\varphi f(k)\frac{1}{D^+(\omega(k))}\times\int\mathbf{dk}'\frac{f(k')a^\dagger(k')}{\omega(k')-\omega(k)-i\eta}. \tag{15}$$

This in conjunction with Eq. (8) gives $\tilde{a}^\dagger(k,\omega)$ explicitly, thus the Heisenberg field.

III. IN-FIELD FOR THE LIGHT MESONS

With the aid of the retarded Green's function,

$$\Delta_R(t,\omega(k))=i\theta(t)e^{i\omega(k)t},$$

defined to satisfy

$$(i(d/dt)+\omega(k))\Delta_R(t-t',\omega(k))=-\delta(t-t'),$$

and letting

$$i(d/dt)a^\dagger(k,t)+\omega(k)a^\dagger(k,t)\equiv j^\dagger(k,t),$$

the in-field is

$$a_{\text{in}}^\dagger(k,t)=a^\dagger(k,t)+\int\Delta_R(t-t',\omega(k))j^+(k,t)e^{-\epsilon(t-t')}dt'. \tag{16}$$

This expression differs from that in Ref. 2 by the inclusion of the damping factor under the integral sign, which was necessary to yield a finite integral.[4] The second term integrated by parts yields

$$a_{\text{in}}^\dagger(k,t)=\epsilon\exp[i(\omega(k)+i\epsilon)t]\times\int_{-\infty}^{t}\exp[-i(\omega(k)+i\epsilon)t']a^\dagger(k,t')dt'. \tag{17}$$

By using Eq. (6) in (17), it follows that

$$a_{\text{in}}^\dagger(k,t)=-i\epsilon\int d\omega\frac{\tilde{a}^\dagger(k,\omega)}{\omega-\omega(k)-i\epsilon}e^{i\omega t}, \tag{18}$$

and

$$i\frac{d}{dt}a_{\text{in}}^\dagger(k,t)+\omega(k)a_{\text{in}}^\dagger(k,t)=i\epsilon(a^\dagger(k,t)-a_{\text{in}}^\dagger(k,t)).$$

Making use of Eq. (7) in (18) and also using (10) and

[4] G. Källén, *Brandeis Summer Institute Lectures in Theoretical Physics* (W. A. Benjamin Inc., New York, 1962), Vol. 1, p. 163.

(12) will give

$$a_{\rm in}^\dagger(k,0)\equiv a_{\rm in}^\dagger(k)=a^\dagger(k)-\lambda\varphi^\dagger\varphi f(k)\frac{1}{D^+(\omega(k))}\times\int d\mathbf{k}'\frac{f(k')a^\dagger(k')}{\omega(k')-\omega(k)-i\epsilon}. \quad (19)$$

It is readily verified that $a_{\rm in}^\dagger(k)\varphi^\dagger|0\rangle$ is identical with the incoming scattering state that results from the interaction of a light boson of momentum $\mathbf{k}$ with a heavy boson, $\varphi^\dagger|0\rangle$. It is perhaps of interest to note that the limit $\epsilon\to 0$, in Eq. (19), cannot be performed in any simple way: for example, the usual interpretation of the denominator in terms of a principal value and a δ-function is not possible because the $a^\dagger(k)$'s are linearly independent and hence the usual cancellations that must occur in order that a principal value integral may be finite will not take place. When the matrix element of $a_{\rm in}^\dagger(k)$ is evaluated, $a^\dagger(k')$ is replaced by a c-number function which may give results independent of ϵ. Of course these remarks do not apply to D^+.

Using Eqs. (1), (2), and (19), it follows that

$$[a_{\rm in}(k),a_{\rm in}^\dagger(k')]=\delta(k-k'),$$
$$[H,a_{\rm in}^\dagger(k)]=\omega(k)a_{\rm in}^\dagger(k)+i\epsilon(a_{\rm in}^\dagger(k)-a^\dagger(k)). \quad (20)$$

Since $\epsilon\ll\mu$ it follows that $a_{\rm in}^\dagger(k)$ is the creation operator for an incoming scattering state of energy almost equal to $\omega(k)$.[4a]

For the purpose of expressing the Hamiltonian in terms of the in-fields an analytic representation of $1/D(z)$ will now be derived. Since $D(z)$ is an R function it can have zeros on the real axis below μ for $\lambda<0$. Since $\varphi^\dagger\varphi$ has integral values $D(z)$ can be expressed as

$$D(z)=\sum_n P_nD_n(z),$$

where P_n is the n heavy meson projection operator and

$$D_n(z)=1+\lambda n\int\frac{f(k)^2d\mathbf{k}}{\omega(k)-z}. \quad (21)$$

Letting μ_n be the zero of $D_n(z)$,

$$D_n(\mu_n)=0, \quad (22)$$

it follows that for $z\to\mu_n$

$$\frac{1}{D(z)}\to\frac{r_n}{z-\mu_n}P_n. \quad (23)$$

From Eq. (12), for $\omega>\mu$,

$$\frac{1}{D^+(\omega)}-\frac{1}{D^-(\omega)}=\frac{-2\pi i\lambda\varphi^\dagger\varphi}{|D^+(\omega)|^2}\int f(k)^2\delta(\omega(k)-\omega)d\mathbf{k},$$

and since $1/D(z)$ analytic except for cuts and poles

$$\frac{1}{D(z)}=1-\lambda\varphi^\dagger\varphi\int d\mathbf{k}\frac{f(k)^2}{|D^+(\omega)|^2}\frac{1}{\omega(k)-z}+P(z), \quad (24)$$

where

$$P(z)=\sum_n[r_n/(z-\mu_n)]P_n. \quad (25)$$

Using Eqs. (19), (24), and (25), it follows that

$$\int d\mathbf{k}\omega(k)a_{\rm in}^\dagger(k)a_{\rm in}(k)=\int\omega(k)a^\dagger(k)a(k)d\mathbf{k}+\lambda\varphi^\dagger\varphi G^\dagger G-\lambda\varphi^\dagger\varphi\sum_n r_n\mu_nA_n^\dagger A_nP_n, \quad (26)$$

where terms proportional to ϵ have been omitted and

$$A_n^\dagger=\int\frac{f(k)a^\dagger(k)}{\omega(k)-\mu_n}. \quad (27)$$

Combining this result with Eq. (1) it follows that

$$H=M\varphi^\dagger\varphi+\int d\mathbf{k}\omega(k)a_{\rm in}^\dagger(k)a_{\rm in}(k)+\lambda\varphi^\dagger\varphi\sum_n r_n\mu_nA_n^\dagger A_nP_n. \quad (28)$$

As will now be shown, $P_nA_n^\dagger$ is a creation operator for a light boson bound to n heavy bosons. From Eqs. (19) and (27) it follows that

$$[a_{\rm in}(k),P_nA_n^\dagger]=[a_{\rm in}(k),P_nA_n]=0. \quad (29)$$

Note that

$$[A_n,A_n^\dagger]=\int d\mathbf{k}\frac{f(k)^2}{(\omega(k)-\mu_n)^2}, \quad (30)$$

whereas from Eqs. (21) and (23) it follows that

$$\int\frac{f(k)^2d\mathbf{k}}{(\omega(k)-\mu_n)^2}=\frac{1}{\lambda nr_n}, \quad (31)$$

thus combining Eqs. (28)–(31),

$$[H,P_nA_n^\dagger]=\mu_nP_nA_n^\dagger. \quad (32)$$

It is clear that $P_nA_n^\dagger$ is different from zero in the subspace of n-heavy bosons only. With the aid of Eqs. (19), (24), and (27), it is readily shown that

$$G^\dagger=\int d\mathbf{k}\frac{a_{\rm in}^\dagger(k)}{D^-(\omega(k))}-\sum_n A_n^\dagger r_nP_n. \quad (33)$$

By comparison of Eqs. (4) and (17) it follows that

$$e^{+iHt}a_{\rm in}^\dagger(k)e^{-iHt}=a_{\rm in}^\dagger(k,t),$$

[4a] In obtaining the commutation of Eq. (20) we have assumed that $a_{\rm in}^\dagger(k)$ is independent of ϵ, thus it is only true for smoothed fields.

and combining this with Eqs. (32) and (33) it gives

$$G^\dagger(t)=e^{iHt}G^\dagger e^{-iHt}$$
$$=\int d\mathbf{k}\frac{a_{\rm in}^\dagger(k,t)}{D^-(\omega(k))}-\sum_n e^{+i\mu_n t}A_n^\dagger r_n P_n. \quad (34)$$

This result will be useful in constructing the in-field for $\varphi^\dagger$.

IV. TRANSFORMATION FROM FREE TO IN-FIELDS

In this section the transformation from the free to the in-fields will be further examined since it turns out to be useful in constructing the in-fields for the heavy bosons in the next section.

Let

$$a_{\rm in}^\dagger(k)=a^\dagger(k')(1+\alpha)_{k'k}, \quad (35)$$

where repeated indices imply integration, this by comparison with Eq. (19) gives

$$\alpha_{k'k}=-\lambda\varphi^\dagger\varphi\frac{f(k)f(k')}{D^\dagger(\omega(k))(\omega(k')-\omega(k)-i\epsilon)}. \quad (36)$$

Note that

$$a_{\rm in}(k)=(1+\alpha^\dagger)_{kk'}a(k').$$

From the commutation relations of the in-fields and Eq. (36) it follows that

$$(1+\alpha^\dagger)(1+\alpha)=1,$$
$$[(1+\alpha)(1+\alpha^\dagger)]_{k'k''}=\delta(k'-k'')-\lambda\varphi^\dagger\varphi\sum_n\frac{f(k')f(k'')r_nP_n}{(\omega(k')-\mu_n)(\omega(k'')-\mu_n)}, \quad (37)$$

thus $(1+\alpha)$ is an isometric matrix when the system has bound states.[5] This is closely related to the corresponding property of the Møller matrix.

It will now be shown that the transformation which leads from $a^\dagger(k)$ to $a_{\rm in}^\dagger(k)$ is not unitary when bound states are present. Let us first construct an operator U such that

$$Ua^\dagger(k)=a_{\rm in}^\dagger(k)U, \quad (38)$$

which gives

$$[U,a^\dagger(k)]=a^\dagger(k')\alpha_{k'k}U \quad (39)$$

after using Eq. (35); U and $U^\dagger$ are found by inspection to be

$$U=1+a^\dagger(k')a(k'')\alpha_{k'k''}+(1/2!)a^\dagger(k')a^\dagger(k'')a(k''')\times a(k'''')\alpha_{k'k'''}\alpha_{k''k''''}+\cdots, \quad (40)$$
$$U^\dagger=1+a^\dagger(k')a(k'')(\alpha^\dagger)_{k'k''}+(1/2!)a^\dagger(k')a^\dagger(k'')\times a(k''')a(k'''')(\alpha^\dagger)_{k'k'''}(\alpha^\dagger)_{k''k''''}+\cdots.$$

From this it follows that

$$[U^\dagger,a^\dagger(k)]=a^\dagger(k')(\alpha^\dagger)_{k'k}U^\dagger. \quad (41)$$

From Eqs. (37), (39), and (41) it is found that

$$[U^\dagger U,a^\dagger(k)]=0, \quad (42)$$

thus[6]

$$U^\dagger U=1. \quad (43)$$

By the same procedure it follows that

$$[UU^\dagger,a^\dagger(k)]=-\lambda\varphi^\dagger\varphi f(k)\sum_n\frac{A_n^\dagger r_n}{\omega(k)-\mu_n}P_nUU^\dagger,$$

hence

$$UU^\dagger=1-\lambda\varphi^\dagger\varphi\sum_n A_n^\dagger A_n r_n P_n+\frac{(\lambda\varphi^\dagger\varphi)^2}{2!}\sum_n A_n^\dagger A_n^\dagger A_n A_n r_n^2 P_n-\cdots. \quad (44)$$

Thus, as stated above, U is not unitary and as will be made clear below $UU^\dagger$ projects out all bound states. In the absence of bound states (for example, $\lambda>0$)

$$a_{\rm in}^\dagger(k)=Ua^\dagger(k)U^\dagger,$$

which follows from Eq. (38). In general

$$a^\dagger(k)=U^\dagger a_{\rm in}^\dagger(k)U.$$

From Eqs. (35) and (41) it is found that

$$U^\dagger a_{\rm in}^\dagger(k)=a^\dagger(k)U^\dagger,$$
$$a_{\rm in}(k)U=Ua(k). \quad (45)$$

By taking the Hermitian conjugate of Eq. (41) and using (27) the following, to be used in the next section, are obtained

$$[P_nA_n,U]=-UA_nP_n,$$

consequently

$$P_nA_nU=U^\dagger A_n^\dagger P_n=0. \quad (46)$$

V. IN-FIELD FOR HEAVY BOSON AND BOUND STATES

In this section an in-field for the heavy boson is defined by analogy to Eq. (38); a more direct approach using Eq. (17) is left to the Appendix. Let

$$U\varphi^\dagger=\varphi_{\rm in}^\dagger U,$$

a solution to this equation is[7]

$$\varphi_{\rm in}^\dagger=U\varphi^\dagger U^\dagger. \quad (47)$$

From Eq. (38) and (45)

$$[a_{\rm in}^\dagger(k),\varphi_{\rm in}^\dagger]=[a_{\rm in}(k),\varphi_{\rm in}^\dagger]=0. \quad (48)$$

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[5] E. C. G. Sudarshan, *Brandeis Summer Institute Lectures in Theoretical Physics* (W. A. Benjamin Inc., New York, 1961), Vol. 2, p. 204.

[6] Additive terms to the right-hand side of Eq. (45) that depend on $\varphi^\dagger\varphi$ are excluded by the observation that $U^\dagger U(\varphi^\dagger)^n|0\rangle=(\varphi^\dagger)^n 0\rangle$, as follows from Eq. (40).

[7] From Eqs. (44) and (45), $Ua^\dagger(k)U^\dagger=UU^\dagger a_{\rm in}^\dagger(k)=a_{\rm in}^\dagger(k)UU^\dagger\neq a_{\rm in}^\dagger(k)$, whereas in a subspace that is free of bound states $Ua^\dagger(k)U^\dagger=a_{\rm in}^\dagger(k)$.

From Eqs. (2) and (43)

$$[\varphi^\dagger\varphi,\varphi_{\rm in}{}^\dagger]=\varphi_{\rm in}{}^\dagger, \tag{49}$$

$$[\varphi_{\rm in},\varphi_{\rm in}{}^\dagger]=UU^\dagger. \tag{50}$$

Thus one encounters operator valued commutators whose appearance in this model is directly traced to the presence of bound states. When the coupling constant is made less negative μ_n moves to the right and λr_n decreases and $UU^\dagger\to 1$. In the absence of bound states $U^\dagger U=UU^\dagger=1$. As consequence of Eqs. (28), (46), (48), and (49)

$$[H,\varphi_{\rm in}{}^\dagger]=M\varphi_{\rm in}{}^\dagger. \tag{51}$$

If instead of the Hamiltonian of Eq. (28) that of Eq. (1) was used then on the right-hand side of (51) terms proportional to ϵ would also be present.

Fields that create bound states will now be examined; let

$$\varphi_B{}^\dagger(1)\equiv A_1{}^\dagger P_1\varphi_{\rm in}{}^\dagger \tag{52}$$

and note that

$$[\varphi_B(1),a_{\rm in}{}^\dagger(k)]=[\varphi_B(1),a_{\rm in}(k)]=0$$

as a consequence of Eqs. (29) and (48). From Eqs. (28), (30), and (31), it follows that

$$[H,\varphi_B{}^\dagger(1)]=(M+\mu_1)\varphi_B{}^\dagger(1), \tag{53}$$

whereas from Eqs. (46), (47), and (52),

$$[\varphi_B(1),\varphi_{\rm in}]=[\varphi_B(1),\varphi_{\rm in}{}^\dagger]=0. \tag{54}$$

$[\varphi_B(1),\varphi_B{}^\dagger(1)]$ is again an operator since the first and second term of the commutator are different from zero in subspaces having different numbers of heavy bosons.

It is clear from Eq. (52) that

$$\varphi_B{}^\dagger(1)\varphi_B{}^\dagger(1)=0. \tag{55}$$

This relation has a rather simple interpretation and will be discussed in the last section.

In the same way the asymptotic field for a light boson bound to n heavy bosons is

$$\varphi_B{}^\dagger(n)=A_n{}^\dagger P_n(\varphi_{\rm in}{}^\dagger)^n \tag{56}$$

and

$$[H,\varphi_B{}^\dagger(n)]=(nM+\mu_n)\varphi_B{}^\dagger(n).$$

From Eq. (46) and (56) it follows that

$$\varphi_B(n)UU^\dagger=0, \tag{57}$$

this justifies the earlier statement that $UU^\dagger$ projects out all bound states.

VI. ASYMPTOTIC LIMIT OF HEISENBERG FIELDS

In this section the in-field will be compared with the $t\to-\infty$ limit of Heisenberg fields. The Fourier transform of the latter is already given in Eq. (7) where for $\omega>\mu$

$$\tilde{G}^\dagger(\omega)=\frac{1}{D^-(\omega)}\int f(k)l(k)\delta(\omega(k)-\omega)\mathbf{dk}, \tag{58}$$

after using Eq. (8). For $\omega<\mu$ combining Eqs. (12), (14), (24), and (27)

$$\begin{aligned}\tilde{G}^\dagger(\omega)&=(1/2\pi i)(\mathcal{L}^+(\omega)-\mathcal{L}^-(\omega))\\&=\frac{1}{2\pi i}\int\mathbf{dk}\frac{f(k)a^\dagger(k)}{\omega(k)-\omega}\left(\frac{1}{D^+(\omega)}-\frac{1}{D^-(\omega)}\right)\\&=-\int\mathbf{dk}\frac{f(k)a^\dagger(k)}{\omega(k)-\omega}\sum_n r_nP_n\delta(\omega-\mu_n)\\&=-\sum_n A_n{}^\dagger r_nP_n\delta(\omega-\mu_n).\end{aligned} \tag{59}$$

By comparison with Eq. (7) this gives

$$a^\dagger(k,t)=\lambda\varphi^\dagger\varphi f(k)\left\{\sum_n\frac{A_n{}^\dagger r_nP_n}{\omega(k)-\mu_n}e^{i\mu_nt}+\int_\mu^\infty\frac{G^\dagger(\omega)e^{i\omega t}d\omega}{\omega-\omega(k)-i\eta}\right\}+l(k)e^{i\omega(k)t}. \tag{60}$$

With the aid of Eq. (58) the second term in the parenthesis above is equal to

$$\int_\mu^\infty\frac{f(k')l(k')\mathbf{dk}'}{D^-(\omega(k'))(\omega(k')-\omega(k)-i\eta)}e^{i\omega(k')t}. \tag{61}$$

If the numerator in the integrand is treated as a continuous function, then as $t\to-\infty$ this integral can readily be shown to vanish.[8] Since $l(k')$ is not a continuous function, only matrix elements of the Heisenberg field lead to the disappearance of the second term of Eq. (60). Thus the remaining terms oscillate with discrete frequencies μ_n and $\omega(k)$. The last term of Eq. (60) is proportional to $a^\dagger(k)_{\rm in}$ and asymptotically creates mesons of energy $\omega(k)$. In the first term the P_n's can be removed by expressing them as infinite products of $\varphi^\dagger\varphi$. This term is different from zero in a subspace containing at least one heavy boson and there acts as a creation operator [see Eq. (32)]. However, since $\mu_n<\mu$ it has no particle interpretation. Both of these terms can be obtained by a small modification of Eq. (17) which then may be taken as the definition of an eigenfield. Consider

$$\epsilon e^{i(E+i\epsilon)t}\int_{-\infty}^t e^{-i(E+i\epsilon)t'}a^\dagger(k,t)dt'. \tag{62}$$

This expression is an eigenvalue equation for that E that yields a finite result for (62) as $\epsilon\to0$. For $E=\omega(k)$ it gives $a^\dagger(k)_{\rm in}e^{i\omega(k)t}$, whereas for $E=\mu_n$, the substitution of Eqs. (7) and (59) into (62) yields

$$\lambda\varphi^\dagger\varphi f(k)\{A_n{}^\dagger r_nP_n/[\omega(k)-\mu_n]\}e^{i\mu_nt}.$$

The application of (62) to any eigenfunction of energy

[8] See H. Ezawa, Ref. 3, for instance.

shows formally, using Eq. (4), that it will result in a state whose energy is increased by E. For arbitrary E the norm of the resulting state is zero. Thus (62) defines an eigenfield for those values of E that make it nonzero. Therefore, it has been shown that as $t\to-\infty$ $a^\dagger(k,t)$ becomes a superposition of eigenfields. Those whose energies are larger than μ are the in-fields.

The Heisenberg and in-fields for the heavy bosons are discussed in the Appendix.

VII. DISCUSSION

The in-fields given in previous sections can now be used to construct the full S matrix for channels with an arbitrary number of particles. As discussed above the implied limit in the definition of in-fields cannot be performed before taking their matrix elements. However, this is not of any consequence for obtaining the S matrix.

The eigenfields defined through Eq. (62) are a natural generalization of Eq. (17) which was used to obtain $a_{\rm in}^\dagger(k)$. From this point of view the large time behavior of the Heisenberg field $a^\dagger(k,t)$ is determined by the superposition of two eigenfields with $E=\omega(k)$ and μ_n, respectively. The latter term vanishes in the subspace free of heavy bosons. When there are bound states, the occurrence of eigenfields that vanish in a given subspace is expected in any static model.

Some of the commutators of the in-field are operators. This was a very simple explanation which could have been anticipated. Any, creation of destruction, in-field heavy boson operator when applied to the subspace containing a bound state of energy E_B must yield zero for otherwise the energy of this new state would be $E_B\pm M$. However, in this model since all bound states and heavy bosons are localized they must have a nonzero energy of interaction; thus to avoid a contradiction the statement above must be valid. It should be noted by comparison with Eq. (50) that its right-hand side is zero in the subspace of bound states and equal to one elsewhere. In the rest of the Hilbert space that is free of bound states the application of $\varphi_{\rm in}^\dagger$ gives a result different from zero since in those states the light bosons are not well localized and the energy is additive. The explanation of Eq. (55) is the same; two bound states in the same place have a total energy that is not additive. If the right-hand side of Eq. (55) were different from zero one would reach a contradiction. Note that in this model two heavy bosons have additive energy since they have no meson cloud. The commutators of the bound-state fields are also operators. Thus the appearance of operator commutation relations is model independent and is due to localizability.

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APPENDIX

Consider the expression

$$U\varphi^\dagger U^\dagger=\varphi'^\dagger \tag{A1}$$

which is equivalent to

$$\varphi^\dagger=U^\dagger\varphi'^\dagger U. \tag{A2}$$

The purpose of the following development is to show that $\varphi'^\dagger=\varphi_{\rm in}^\dagger$, where $\varphi_{\rm in}^\dagger$ is defined through Eq. (17). Using Eq. (51) the Heisenberg field at time t is

$$\varphi^\dagger(t)=U^\dagger(t)\varphi'^\dagger U(t)e^{iMt}, \tag{A3}$$

where

$$U^\dagger(t)=e^{iHt}U^\dagger e^{-iHt}. \tag{A4}$$

In this Appendix the Hamiltonian of Eqs. (1) and (28) will be used interchangeably. From Eqs. (1), (28), and (40),

$$[H,U^\dagger]=\lambda G^\dagger\varphi^\dagger\varphi U^\dagger\int f(k)a_{\rm in}(k)\mathbf{dk},$$

thus

$$-i\frac{d}{dt}U^\dagger(t)=\lambda\varphi^\dagger\varphi G^\dagger(t)U^\dagger(t)\int f(k)a_{\rm in}(k)e^{-i\omega(k)t}\mathbf{dk}.$$

Letting $U^\dagger(0)=U^\dagger$ the solution of the above equation is

$$U^\dagger(t)=U^\dagger+i\lambda\varphi^\dagger\varphi\int_0^t dt'G^\dagger(t')U^\dagger G_{\rm in}(t') +\frac{(i\lambda\varphi^\dagger\varphi)^2}{2!}\int_0^t\int_0^t dt'dt''G^\dagger(t')G^\dagger(t'')U^\dagger \times G_{\rm in}(t')G_{\rm in}(t'')+\cdots,$$

where

$$G_{\rm in}(t)=\int f(k)a_{\rm in}(k)e^{-i\omega(k)t}\mathbf{dk}.$$

From Eq. (33)

$$G^\dagger(t)=\int \mathbf{dk}\frac{a_{\rm in}^\dagger(k)}{D^-(\omega(k))}e^{i\omega(k)t}-\sum_n A_n^\dagger r_nP_ne^{i\mu_nt}. \tag{A5}$$

Using Eq. (38) in $U^\dagger(t)$ given above

$$U^\dagger(t)=\left\{1+i\lambda\varphi^\dagger\varphi\int_0^t dt'G^\dagger(t')G_0(t') +\frac{(i\lambda\varphi^\dagger\varphi)^2}{2!}\int_0^t\int_0^t dt'dt''G^\dagger(t')G^\dagger(t'') \times G_0(t')G_0(t'')+\cdots\right\}U^\dagger, \tag{A6}$$

where

$$G_0(t)=\int \mathbf{dk}f(k)a(k)e^{-i\omega(k)t}. \tag{A7}$$

After using (A5) and (A7) any term of (A6), except the first, is of the form

$$\int_0^t G^\dagger(t')Q(t)G_0(t')dt'$$
$$-i\left(\int \frac{a^\dagger{}_{\text{in}}(k)f(k)}{D^-(\omega(k))}Q(t)a(k')f(k') \times\frac{\exp[i(\omega(k)-\omega(k'))t]-1}{\omega(k)-\omega(k')}\mathbf{dk}\mathbf{dk}' -\int\sum_n A_n{}^\dagger r_n P_n Q(t)a(k')f(k') \times\frac{\exp[i(\mu_n-\omega(k'))t]-1}{\mu_n-\omega(k')}\mathbf{dk}'\right).$$

Using above the identity

$$\lim_{t\to-\infty} P\int_{-\alpha}^{\infty} e^{itx}\frac{g(x)}{x}dx=-\pi i g(0)\,,\quad \alpha>0\,,$$

where $g(x)$ is a smooth function, it follows that

$$\lim_{t\to-\infty}\int_0^t G^\dagger(t')Q(t)G_0(t')dt'$$
$$=i\left(\int\frac{a_{\text{in}}{}^\dagger(k)f(k)}{D^-(\omega(k))}\lim_{t\to-\infty}Q(t)\frac{a(k')f(k')}{\omega(k)-\omega(k')-i\epsilon}\mathbf{dk}\mathbf{dk}' +\sum_n r_nA_n{}^\dagger P_n\lim_{t\to-\infty}Q(t)A_n\right). \quad \text{(A8)}$$

This relation holds for matrix elements that replace $a(k')$ and $a_{\text{in}}{}^\dagger(k')$ by smooth functions. Using Eqs. (35) and (37) the above simplifies and gives

$$\lim_{t\to-\infty} i\lambda\varphi^\dagger\varphi\int_0^t G(t')Q(t)G_0(t')dt' =\int a^\dagger(k'')\alpha_{k''k'}\lim_{t\to-\infty}Q(t)a(k')\mathbf{dk}'\mathbf{dk}''.$$

Repeating this argument for $U(t)$ given in (A6)

$$\lim_{t\to-\infty}U^\dagger(t)=UU^\dagger. \quad \text{(A9)}$$

Comparing (A1), (A3) with (A9) yields

$$\lim_{t\to-\infty}\varphi^\dagger(t)\to U\varphi^\dagger U^\dagger\,, \quad \text{(A10)}$$

in the sense of weak operator convergence.

The construction of the in-field is greatly simplified by the following relation,

$$\lim_{\epsilon\to0}\epsilon\int_{-\infty}^0 e^{\epsilon t'}\frac{1-e^{ixt'}}{x}\frac{1-e^{iyt'}}{y}\frac{1-e^{izt'}}{z}\cdots =\frac{1}{x-i\epsilon}\frac{1}{y-i\epsilon}\frac{1}{z-i\epsilon}\cdots\,, \quad \text{(A11)}$$

which is only valid in an integral sense over smooth functions. In applying (A11) below to calculate the in-field the smoothness of the rest of the integrand can be justified only after taking matrix elements.

From Eq. (17), for $t=0$

$$\varphi_{\text{in}}{}^\dagger=\epsilon\int_{-\infty}^0 e^{-i(M+i\epsilon)t'}\varphi^\dagger(t')dt'\,; \quad \text{(A12)}$$

this, after substitution from (A3), becomes

$$\varphi_{\text{in}}{}^\dagger=\epsilon\int_{-\infty}^0 e^{\epsilon t'}U^\dagger(t')\varphi'^\dagger U(t')dt'. \quad \text{(A13)}$$

Since every term of $U^\dagger(t')$ and $U(t')$ is of the form

$$(1-e^{ixt'})/x\,,$$

it follows, from (A11) and (A12), that

$$\varphi_{\text{in}}{}^\dagger=\epsilon\int_{-\infty}^0 e^{\epsilon t'}U^\dagger(t')\varphi'^\dagger\epsilon\int_{-\infty}^0 e^{\epsilon t''}U(t'')dt''. \quad \text{(A14)}$$

By comparison of (A6), (A8), (A9), and (A11),

$$\lim_{t\to-\infty}U^\dagger(t')=\epsilon\int_{-\infty}^0 e^{\epsilon t'}U^\dagger(t')=UU^\dagger\,, \quad \text{(A15)}$$

in the sense of weak operator convergence. Combining (A1) and (A15), we obtain

$$\varphi_{\text{in}}{}^\dagger=UU^\dagger\varphi'^\dagger UU^\dagger=\varphi'^\dagger. \quad \text{Q.E.D.}$$

Thus

$$U\varphi^\dagger U^\dagger=\varphi_{\text{in}}{}^\dagger=\lim_{t\to-\infty}\varphi^\dagger(t)$$

under the restrictions stated above.